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Short Communication

ON DEPENDENT ELEMENTS AND FREE ACTION OF DERIVATIONS IN SEMIPRIME Γ-RINGS

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ABSTRACT

In this paper we characterize a dependent element of a derivation on a semiprime Γ-ring M. It is shown that the dependent elements of a derivation d of a semiprime Γ-rings M are central, and it is proved that if a is a dependent element of a derivation d of M, then there exist ideals U and V of the semiprime Γ-ring M such that (i)U⊕V is an essential ideal of M. (ii) d =0 on U, and d(V) ⊆V and (iii) the derivation d of the semiprime Γ-ring M is free action on V. Furthermore, some of results of free action for several mappings in prime and semiprime Γ-rings are given.

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Introduction

The notion of a Γ-ring was introduced by Nobusawa (1964) and generalized by Barnes (1966) ∈as follows: Let M and Γ be two additive abelian groups. If for all x, y, z M and α, β∈ Γ, the following conditions are satisfied

(1) $x\alpha y \in M$,
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring (in the sense of Barnes, 1966). If these conditions are strengthened to

(1`) $x\alpha y \in M$, and $\alpha x\beta \in \Gamma$,
(2`) same as (2),
(3`) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$,
(4`) $x\alpha y = 0$ for all $x, y \in M$ implies $\alpha = 0$,

we then have M a Γ-ring in the sense of Nobusawa (1964). We may note that it follows from (1)→(3) that $0\alpha x = x0y = 0\alpha x = 0$, for all x, y∈ M and α∈ Γ.

Recall that a Γ-ring M is called a prime if for any two elements x, y ∈ M, xΓMΓy = 0 implies either x = 0 or y = 0, and M is called semiprime if xΓMΓx = 0 with x ∈ M implies x = 0. Note that every prime Γ-ring is obviously semiprime. An additive mapping d: M → M is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all x, y ∈ M and α ∈ Γ. An additive subgroup U of M is called a left (right) ideal of M if MΓU⊆U (UΓM⊆U). If U is both left and right ideal of M, then we say U is an ideal of M. An ideal of M is said to be essential if it has non zero intersection with any non zero ideal of M. For a subset U of M, $Ann_l(U) = \{a \in M | a\Gamma U = \langle 0\rangle \}$is called the left annihilator of U. A right annihilator $Ann_r(U)$ can be defined similarly. It is known that the right and left annihilators of an ideal U of a semiprime Γ-ring M coincide, it will be denoted by Ann(U) (Ztürk and Yazarli, 2007). It is easy to show that U∩Ann(U) = {0} and U⊕Ann(U) is an essential of the Γ-ring M. Following (Pual and Sabur, 2010) an element x of M is called nilpotent if for some γ∈ Γ, there exists a positive integer n = n(γ) such that $(x\gamma)^n x = 0$ and an ideal U of a Γ-ring M is called nilpotent if $(U\Gamma)^n U = 0$, where n is the least positive integer. Furthermore, M is said to be a commutative Γ-ring if xαy = yαx for all x, y ∈ M and α ∈ Γ. The set Z(M) = {x ∈M; xαy = yαx for all x, y ∈ M and α ∈ Γ} is called the center of M. The commutator xαy – yαx will be denoted by $[x, y]_\alpha$. We will use for all x, y, z ∈ M and α, β ∈ Γ, the basic commutator identities:

$[x\alpha y, z]_\beta = x\alpha[y, z]_\beta + [x, z]_\beta\alpha y + x[\alpha, \beta]_z y$, and
$[x, y\alpha z]_\beta = y\alpha[x, z]_\beta + [x, y]_\beta\alpha z + y[\beta, \alpha]_x z$,

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Throughout this paper, consider the following assumption $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta, \in \Gamma$ and it will represented by (*). According to the assumption (*), the above two identities reduce to $[x\alpha y, z]_\beta = x\alpha[y, z]_\beta + [x, z]_\beta\alpha y$, and
$[x, y\alpha z]_\beta = y\alpha[x, z]_\beta + [x, y]_\beta\alpha z$.

Laradji and Thaheem in (Laradji and Thaheem, 1998) defined the dependent element of a mapping f as follows: An element $a\in R$ is said to be a dependent element of a mapping $f: R \to R$ if
$f(x)a = ax$ for all $x \in R$,

They studied the dependent elements of endomorphisms of semiprime rings and Vukman and Kosi-Ulbl in (Vukman and Kosi-Ulbl, 2004) studied dependent elements of various mappings related to derivations, automorphisms and generalized derivations on prime and semiprime rings. Several other authors have studied dependent elements in prime and semiprime rings (Faisal and Muhammad, 2009; Muhammad and Mohammad, 2008a; Hentzel *et al.,* 2011; Mohammad, and Muhammad, 2008b). Furthermore, a mapping $f: R \to R$ is said to be a free action on M if zero is the only dependent element of f.

In this paper analogous in (Laradji and Thaheem, 1998) we define a dependent element of a mapping f on a semiprime Γ-rings M as follows: An element $a \in M$ is said to be dependent element of a mapping $f: M \to M$ if $f(x)\alpha a = a\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$. A mapping $f: M \to M$ is said to be free action on M if the only dependent element of f is zero. We investigate some properties and give some results of free action for mappings related to derivations on prime and semiprime Γ-rings M. For a mapping $f: M \to M$, D(f) denotes the collection of all dependent elements of f.

1. Dependent elements on derivations

For proving the main results, we start by the following theorem:

Theorem 2.1. Let M be a semiprime Γ-ring satisfying (*) and $d: M \to M$ be a derivation of M. If $a\in D(d)$, then $a\in Z(M)$.

Proof. Suppose that $a \in D(d)$, that is
$$d(x)\alpha a = a\alpha x, \text{ for all } x \in M \text{ and } \alpha\in \Gamma. \qquad (2.1)$$

Replacing x by $x\gamma y$ in (2.1), we get
$$d(x)\gamma y\alpha a + x\gamma d(y)\alpha a = a\alpha x\gamma y, \text{ for all } x, y\in M \text{ and } \alpha, \gamma \in \Gamma. \qquad (2.2)$$

Using (2.1) we get
$$d(x)\gamma y\alpha a = [a, x]_\alpha\gamma y, \text{ for all } x, y\in M \text{ and } \alpha, \gamma \in \Gamma. \qquad (2.3)$$

Hence we obtain
$$d(x)\gamma y\alpha a\beta z = [a, x]_\alpha\gamma y\beta z, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma \in \Gamma. \qquad (2.4)$$

Replacing y by $y\beta z$ in (2.3), we obtain
$$d(x)\gamma y\beta z\alpha a = [a, x]_\alpha\gamma y\beta z, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma \in \Gamma. \qquad (2.5)$$

Subtracting (2.5) from (2.4), we have
$d(x)\gamma y\beta[a, z]_\alpha = 0$, for all $x, y, z\in M$ and $\alpha, \beta, \gamma \in \Gamma$.

Replacing y by $a\delta y$ and using (2.1), we obtain
$$a\gamma x\delta y\beta[a, z]_\alpha = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \qquad (2.6)$$

Hence we obtain
$$z\alpha a\gamma x\delta y\beta [a, z]_\alpha = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \qquad (2.7)$$

Replacing x by $z\alpha x$ in (2.6), we get
$$a\gamma z\alpha x\delta y\beta [a, z]_\alpha = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \qquad (2.8)$$

Subtracting (2.7) from (2.8), we obtain
$$[a, z]_\alpha\gamma x\delta y\beta [a, z]_\alpha = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \qquad (2.9)$$

Hence we obtain $[a, z]_\alpha\gamma x\delta y\beta [a, z]_\alpha\gamma x = 0$, for all $x, y, z\in M$ and $\alpha, \beta, \gamma,\delta \in \Gamma$. By semiprimeness we have $[a, z]_\alpha = 0$, for all $z \in M$ and $\alpha \in \Gamma$. Hence $a\in Z(M)$.

Corollary 2.2. Let M be a semiprime Γ-ring satisfying (*) and $d:M \to M$ be a derivation on M. If $a \in D(d)$, then $d(x)\alpha a =0$.

Proof. Let $a \in D(d)$, by Theorem 2.1 $a\in Z(M)$, that is $d(x)\alpha a = a\alpha x = x\alpha a$, for all $x \in M$ and $\alpha\in \Gamma$.
Replacing x by $x\beta y$, we get
$x\beta y\alpha a = d(x)\beta y\alpha a+ x\beta d(y)\alpha a$
$= d(x)\beta a\alpha y + x\beta d(y)\alpha a = d(x)\beta a\alpha y +x\beta a\alpha y$, for all $x, y\in M$ and $\alpha, \beta, \gamma\in\Gamma$.

Being $a \in Z(M)$ the last relation implies that $d(x)\beta a\alpha y = 0$. By semiprimeness we get $d(x)\alpha a = 0$, for all $x \in M$ and $\alpha \in \Gamma$.

Corollary 2.3. Let M be a semiprime Γ-ring satisfying (*) and d: M → M be a derivation on M. If $a \in D(d)$ then $d(a) = 0$.

Proof. Since $a \in D(d)$, by corollary 2.2 we get
$d(x)\alpha a = 0$, for all $x \in M$
and $\alpha \in \Gamma$. (2.10)

Replacing x by d(x) in (2.10) we get
$d(d(x))\alpha a = 0$, for all $x \in M$
and $\alpha \in \Gamma$. (2.11)

From (2.10), we obtain $d(d(x)\alpha a) = d(0) = 0$, which implies $d(d(x))\alpha a + d(x)\alpha d(a) = 0$, for all $x \in M$ and $\alpha \in \Gamma$. Using (2.11), we get $d(x)\alpha d(a) = 0$, for all $x \in M$ and $\alpha \in \Gamma$. Replacing x by $a\beta x$, we have $d(a)\beta x\alpha d(a) = 0$, for all $x \in M$ and $\alpha, \beta \in \Gamma$. By semiprimeness we get $d(a) = 0$.

Remark 2.4. Let M be a semiprime Γ-ring and U be a right ideal of M, then U is a semiprime subring of M and $Z(U) \subseteq Z(M)$.

Now we prove the main result in this section.

Theorem 2.5. Let M be a semiprime Γ-ring satisfying (*) and d: M → M be a derivation on M. Let a be a dependent element of d. Then there exist ideals U and V of M such that
(i) $U \oplus V$ is an essential ideal of M.
(ii) $d = 0$ on U and $d(V) \subseteq V$.
(iii) $D(d|_V) = \{0\}$, where $d|_V$ is restriction of d on V.

That is d free action on V.

Proof. (i) Since $a \in D(d)$, by Theorem 2.1 we have $a \in Z(M)$. Therefore
$a\alpha m = m\alpha a$, for all $m \in M$ and $\alpha \in \Gamma$,
that is $a\Gamma M = M\Gamma a$, and this implies $a\Gamma M$ is a two sided ideal of M.

Put $U = a\Gamma M$ and $V = Ann(U)$. Since Ann(U) is an ideal of M, then V is an ideal of M, and $U \oplus V$ is essential of M.

(ii) By Corollary 2.2, Corollary 2.3 and Theorem 2.1 we have $d(x)\alpha a = 0$, $d(a) = 0$, and $a \in Z(M)$, for all $x \in M$, $\alpha \in \Gamma$. Thus $d(a\alpha x) = d(a)\alpha x + a\alpha d(x) = d(a)\alpha x + d(x)\alpha a = 0$, for all $x \in M$, $\alpha \in \Gamma$.

$d(x\alpha a) = d(x)\alpha a + x\alpha d(a) = 0$, and $d(x\alpha a\beta y) = d(x)\alpha a\beta y + x\alpha d(a)\beta y + x\alpha a\beta d(y) = x\alpha d(y)\beta a = 0$, for all $x, y \in M$, $\alpha, \beta \in \Gamma$. Hence d= 0 on U.

Now let $d(v) \in d(V)$, for $v \in V = Ann(U)$, thus $v\alpha a = 0$, for all $a \in U$. So $d(v\alpha a) = d(0) = 0$, and $v\alpha d(a) = 0$, since $a \in U$, and d =0 on U. Then $d(v\alpha a) = d(v)\alpha a + v\alpha d(a)$, this implies that $d(v)\alpha a = 0$, that is $d(v) \in Ann(U) = V$.

(iii) Since V is an ideal of M, by Remark 2.4 we have $Z(V) \subseteq Z(M)$. Since $d(V) \subseteq V$ so $d|_V$ is a derivation on V. Now let $a \in V$ be a dependent element of $d|_V$ on V, then by Theorem 2.1 we have $a \in Z(V)$, and by Remark 2.4 we have $Z(V) \subseteq Z(M)$, that is $a \in Z(M)$. By Corollary 2.2 and Corollary 2.3 we have $d|_v(v)\alpha a = 0 = a\alpha d|_v(v)$ and $d|_v(a) = 0$.

Let $x \in M$, so $x\beta v \in V$. Thus $d(x\beta v)\alpha a = 0$, but $d(x\beta v)\alpha a = d(x)\beta v\alpha a + x\beta d(v)\alpha a$, this implies that $d(x)\beta v\alpha a = 0$.

Since $a \in Z(V)$, so $d(x)\beta a\alpha v = 0$, for all $x \in M$, $v \in V$ and $\alpha, \beta \in \Gamma$. By by semiprimeness of V we get $d(x)\alpha a = 0$, and by definition we obtain $a\alpha x = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then by semiprime of M we get $a = 0$. Hence $D(d|_V) = \{0\}$ on V.

2. Free actions of prime and semiprime Γ-rings

In this section we discuss free action for mappings related to derivations and we obtain some results. We start by the following lemma:

Lemma 3.1 (Chakraborty and Paul, 2010. Lemma 2.13). Let M be a 2-torsion free semiprime Γ-ring and suppose that $a, b \in M$. If $a\Gamma m\Gamma b + b\Gamma m\Gamma a = 0$ for all $m \in M$, then $a\Gamma m\Gamma b = b\Gamma m\Gamma a = 0$.

Theorem 3.2 (Pual and Sabur, 2010. Theorem 3.1). Let M be a prime Γ-ring and $U \neq 0$ be a right ideal of M. suppose that $a \in U$, $(a\gamma)^n a = 0$ for a fixed integer n, then M has non-zero nilpotent ideal.

Remark 3.3. Let M be a Γ-ring. If a is nilpotent, then $M\Gamma a$ is a nilpotent ideal of M. Ina semiprime Γ-ring M, if a is nilpotent, then a = 0.

Theorem 3.4. Let M be a semiprime Γ-ring satisfying (*) and d: M $\to$ M be a derivation. Then Φ:M $\to$ M defined as $\Phi(x) = x\beta d(x)$ for all $x \in M$ and $\beta \in \Gamma$ is free action.

Proof: Suppose that $a \in D(\Phi)$, that is
$\Phi(x)\alpha a = a\alpha x$ for all $x \in M$
and $\alpha \in \Gamma$. (3.1)

Equivalently
$x\beta d(x)\alpha a = a\alpha x$, for all $x \in M$
and $\alpha, \beta \in \Gamma$. (3.2)

Linearizing (3.2) with respect x we get
$x\beta d(y)\alpha a + y\beta d(x)\alpha a = 0$, for all x, $\in M$
and $\alpha, \beta \in \Gamma$. (3.3)

Replacing x and y by a, we obtain $2a\beta d(a)\alpha a = 0$, this implies that
$2a\alpha a = 0$, for all $x \in M$
and $\alpha \in \Gamma$. (3.4)

Replacing y by $x\alpha a$ in (3.3) we get
$x\beta d(x)\alpha a\alpha a + x\beta x\alpha d(a)\alpha a + x\alpha a\beta d(x)\alpha a = 0$, for all $x \in M$ and $\alpha \in \Gamma$.

By using (3.2) we get
$a\alpha x\alpha a + x\beta x\alpha d(a)\alpha a + x\alpha a\beta d(x)\alpha a = 0$.

Replacing x by a we obtain $a\alpha a\alpha a + 2a\alpha a\beta d(a)\alpha a = 0$. By using (3.4) we get $a\alpha a\alpha a = 0$, since M is semiprime Γ-ring , then by Remark 3.3 we get a = 0.

Theorem 3.5. Let M be a 2-torsion free semiprime Γ-ring and d: M $\to$ M be a derivation. Then a mapping Φ : M $\to$ M defined as $\Phi(x) = x\alpha d(x) + d(x)\alpha x$ for all $x \in M$ and a fixed $\alpha \in \Gamma$ is free action.

Proof: Suppose that $a \in D(\Phi)$, that is $\Phi(x)\alpha a = a\alpha x$ for all $x \in M$ and $\alpha \in \Gamma$, equivalently
$x\alpha d(x)+ d(x)\alpha x\}\alpha a = a\alpha x$, for all $x \in M$
and $\alpha \in \Gamma$. (3.5)

Linearizing (3.5) with respect x we get
$x\alpha d(y)\alpha a + y\alpha d(x)\alpha a + d(x)\alpha y\alpha a + d(y)\alpha x\alpha a = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$. (3.6)

Replacing x and y by a, and using (3.5) we obtain
$2a\alpha a = 0$. (3.7)

Since M is 2-torsion free semiprime Γ-ring, by Remark 3.3 we get a = 0.

Theorem 3.5. Let M be a prime Γ-ring satisfying (*) and d, g and h be nonzero derivations of M. Then the mapping Φ : M $\to$ M defined as $\Phi(x) = d(g(x)) + h(x)$ for all $x \in M$ is free action.

Proof: Suppose that $a \in D(\Phi)$, that is

$\Phi(x)\alpha a = a\alpha x$ for all $x \in M$
and $\alpha \in \Gamma$. (3.8)

Equivalently
$(d(g(x)) + h(x))\alpha a = a\alpha x$.
Replacing x by $x\beta a$ we get
$(d(g(x\beta a)) + h(x\beta a))\alpha a = a\alpha x\beta a$.

This implies that
$\Phi(x)\beta a\alpha a + x\beta\Phi(a)\alpha a + g(x)\beta d(a)\alpha a + d(x)\beta g(a)\alpha a = a\alpha x\beta a$.

By using (3.8) we obtain
$x\beta a\alpha a + g(x)\beta d(a)\alpha a + d(x)\beta g(a)\alpha a = 0$ (3.9)

Replacing x by $y\gamma x$ and using (3.9) we obtain
$g(y)\gamma x\beta d(a)\alpha a + d(y)\gamma x\beta g(a)\alpha a = 0$, for all $x, y\in M$ and $\alpha, \beta, \gamma \in \Gamma$. (3.10)

Replacing y by a and x by $a\alpha x$ we obtain
$g(a)\gamma a\alpha x\beta d(a)\alpha a + d(a)\gamma a\alpha x\beta g(a)\alpha a = 0$, for all $x, y\in M$ and $\alpha, \beta, \gamma\in \Gamma$.

By using Lemma 3.1 we conclude that $d(a)\gamma a\alpha x\beta g(a)\alpha a = 0$, and by the primness of M, either $d(a)\alpha a = 0$ or $g(a)\alpha a = 0$.

If both are zero we get from (3.9) that is $x\beta a\alpha a = 0$ and by primeness we get $a\alpha a = 0$, then by Remark 3.3, a = 0.

If $d(a)\alpha a = 0$ and $g(a)\alpha a \neq 0$, then the relation (3.10) yields $d(y)\gamma x\beta g(a)\alpha a = 0$. By primness and relation (3.9) a = 0.

If $g(a)\alpha a = 0$ and $d(a)\alpha a \neq 0$, the relation (3.10) yields $g(y)\gamma x\beta d(a)\alpha a = 0$, and by primeness and relation (3.9), a = 0. So for any case we get a = 0, this implies that Φ is free action.

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